5. cvičení - řešení

Příklad 1 (a)

$$\lim_{n\to\infty} \frac{2^n \sqrt[3]{n^3+n^2} - n\sqrt[3]{8^n+1}}{\sqrt[n]{2^{n^2}+1}} = \lim_{n\to\infty} \frac{2^n\left(\sqrt[3]{n^3+n^2} - n\sqrt[3]{1+\frac{1}{8^n}}\right)}{2^n\sqrt[n]{1+\frac{1}{2^{n^2}}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^3+n^2} - n\sqrt[3]{1+\frac{1}{8^n}}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}} \cdot \frac{\sqrt[3]{(n^3+n^2)^2} + \sqrt[3]{n^3+n^2}\cdot n\sqrt[3]{1+\frac{1}{8^n}} + n^2\sqrt[3]{(1+\frac{1}{8^n})^2}}{\sqrt[3]{(n^3+n^2)^2} + \sqrt[3]{n^3+n^2}\cdot n\sqrt[3]{1+\frac{1}{8^n}} + n^2\sqrt[3]{(1+\frac{1}{8^n})^2}} =$$

$$= \lim_{n\to\infty} \frac{n^3+n^2-n^3-\frac{n^3}{8^n}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot n^2\cdot\left(\sqrt[3]{\left(1+\frac{1}{n}\right)^2} + \sqrt[3]{1+\frac{1}{n}}\cdot\sqrt[3]{1+\frac{1}{8^n}} + \sqrt[3]{(1+\frac{1}{8^n})^2}\right)} =$$

$$= \lim_{n\to\infty} \frac{1+\frac{n}{8^n}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot\left(\sqrt[3]{\left(1+\frac{1}{n}\right)^2} + \sqrt[3]{1+\frac{1}{n}}\cdot\sqrt[3]{1+\frac{1}{8^n}} + \sqrt[3]{(1+\frac{1}{8^n})^2}\right)} \overset{\text{VoAL}}{=} \frac{1}{3}$$

Poslední rovnost jsme získali díky opakovénu použití věty o dvou strážnících na výrazy

$$c_n = \sqrt[n]{1+\frac{1}{2^{n^2}}},$$

$$d_n = \sqrt[3]{\left(1+\frac{1}{n}\right)^2} + \sqrt[3]{1+\frac{1}{n}}\cdot\sqrt[3]{1+\frac{1}{8^n}} + \sqrt[3]{(1+\frac{1}{8^n})^2}.$$

Pak máme

$$\sqrt[n]{1} \le c_n \le \sqrt[n]{2}$$

$$\lim_{n\to\infty}\sqrt[n]{1} = \lim_{n\to\infty}\sqrt[n]{2} = 1.$$

Dále máme

$$\sqrt[3]{(1)^2} + \sqrt[3]{1}\cdot\sqrt[3]{1} + \sqrt[3]{(1)^2} \le d_n \le 3\cdot\sqrt[3]{\left(1+\frac{1}{n}\right)^2}$$

$$\lim_{n\to\infty}\sqrt[3]{(1)^2} + \sqrt[3]{1}\cdot\sqrt[3]{1} + \sqrt[3]{(1)^2} = \lim_{n\to\infty} 3\cdot\sqrt[3]{\left(1+\frac{1}{n}\right)^2} \overset{\text{VOAL + o posl. s kladnými členy}}{=} 3$$

Příklad 1 (b)

$$\lim_{n\to\infty}\left(\sqrt{n^2+\sqrt{n}+1}-\sqrt{n^2+2\sqrt{n}+3}\right)\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$=\lim_{n\to\infty}\frac{n^2+\sqrt{n}+1-(n^2+2\sqrt{n}+3)}{\sqrt{n^2+\sqrt{n}+1}+\sqrt{n^2+2\sqrt{n}+3}}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$=\lim_{n\to\infty}\frac{-\sqrt{n}-2}{\sqrt{n^2+\sqrt{n}+1}+\sqrt{n^2+2\sqrt{n}+3}}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$=\lim_{n\to\infty}\frac{\sqrt{n}}{n}\frac{-1-\frac{2}{\sqrt{n}}}{\sqrt{1+\frac{1}{n\sqrt{n}}+\frac{1}{n^2}}+\sqrt{1+\frac{2}{n\sqrt{n}}+\frac{3}{n^2}}}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$\overset{\text{VoAL + o posl. s kladnými členy}}{=}-\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{n}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}\overset{\text{VoAL}}{=}-\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{\lfloor\sqrt{n+2}\rfloor}\lim_{n\to\infty}\frac{\sqrt[n]{n^n+n}}{n}=(*)$$

Platí

$$\sqrt{n}-1\le\lfloor\sqrt{n}\rfloor\le\lfloor\sqrt{n+2}\rfloor\le\sqrt{n+2},$$

a tedy

$$\frac{\sqrt{n}}{\sqrt{n+2}}\le\frac{\sqrt{n}}{\lfloor\sqrt{n+2}\rfloor}\le\frac{\sqrt{n}}{\sqrt{n}-1}.$$

Zároveň

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+2}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{2}{n}}}\overset{\text{VoAL}}{=}1$$

a

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}-1}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{\sqrt{n}}}\overset{\text{AL}}{=}1.$$

Z Věty o dvou policajtech pak platí $\lim_{n\to\infty}\frac{\sqrt{n}}{\lfloor\sqrt{n+2}\rfloor}=1$. Dále platí

$$1=\frac{\sqrt[n]{n^n}}{n}\le\frac{\sqrt[n]{n^n+n}}{n}\le\frac{\sqrt[n]{2n^n}}{n}=\sqrt[n]{2}$$

a

$$\lim_{n\to\infty}1=1=\lim_{n\to\infty}\sqrt[n]{2}.$$

Opět použitím Věty o dvou policajtech dostáváme

$$(*)=-\frac{1}{2}\cdot1\cdot1=-\frac{1}{2}$$

Příklad 1 (c)

$$\lim_{n\to\infty}\left(\sqrt[3]{2n+a}-\sqrt[3]{2n+b}\right)\cdot\sqrt[3]{(n+1)(3n+2)}=$$

$$=\lim_{n\to\infty}\sqrt[3]{(n+1)(3n+2)}\left(\sqrt[3]{2n+a}-\sqrt[3]{2n+b}\right)\cdot\frac{(\sqrt[3]{2n+a})^2+\sqrt[3]{2n+a}\cdot\sqrt[3]{2n+b}+(\sqrt[3]{2n+b})^2}{(\sqrt[3]{2n+a})^2+\sqrt[3]{2n+a}\cdot\sqrt[3]{2n+b}+(\sqrt[3]{2n+b})^2}=$$

$$=\lim_{n\to\infty}\frac{((2n+a)-(2n+b))\cdot\sqrt[3]{(n+1)(3n+2)}}{(\sqrt[3]{2n+a})^2+\sqrt[3]{2n+a}\cdot\sqrt[3]{2n+b}+(\sqrt[3]{2n+b})^2}=$$

$$=\lim_{n\to\infty}\frac{(a-b)n^{\frac{2}{3}}\sqrt[3]{(1+\frac{1}{n})(3+\frac{2}{n})}}{n^{\frac{2}{3}}\left((\sqrt[3]{2+\frac{a}{n}})^2+\sqrt[3]{2+\frac{a}{n}}\cdot\sqrt[3]{2+\frac{b}{n}}+(\sqrt[3]{2+\frac{b}{n}})^2\right)}\overset{\text{VoAL}}{=}\frac{(a-b)\cdot\sqrt[3]{3}}{3\cdot 2^{\frac{2}{3}}}=(a-b)\cdot 6^{-\frac{2}{3}}$$

Použili jsme vzorec
$A^n-B^n=(A-B)(A^{n-1}+A^{n-2}B+A^{n-3}B^2+\cdots+A^2B^{n-3}+AB^{n-2}+B^{n-1})$.
Příklad 1 (d) Pro $k\in\mathbb{N}$ označme $P_{\le k}(n)$ polynom proměnné n stupně nejvýše k.

$$\lim_{n\to\infty}\frac{(n^2+1)^{100}-(n+2)^{200}+400n^{199}}{1+2+\ldots+n^{99}}$$

$$=\lim_{n\to\infty}\frac{n^{200}+100n^{198}+P_{\le 196}(n)-n^{200}-400n^{199}-4\cdot\binom{200}{2}n^{198}-P_{\le 197}(n)+400n^{199}}{\frac{n^{99}(n^{99}+1)}{2}}$$

$$=\lim_{n\to\infty}\frac{n^{198}(100-79600+\frac{P_{\le 197}(n)}{n^{198}})}{n^{198}(\frac{1}{2}+\frac{1}{n^{99}})}=\lim_{n\to\infty}\frac{100-79600+\frac{P_{\le 197}(n)}{n^{198}}}{\frac{1}{2}+\frac{1}{n^{99}}}\overset{\text{VoAL}}{=}-159000.$$

Příklad 1 (e) $\lim_{n\to\infty}n^3\left(\sqrt[3]{n!+3^n}-\sqrt[3]{n!+2^n}\right)$
Použijeme vzorec $a^3-b^3=(a-b)(a^2+ab+b^2)$.

$$\lim_{n\to\infty}n^3\left(\sqrt[3]{n!+3^n}-\sqrt[3]{n!+2^n}\right)=\lim_{n\to\infty}n^3\cdot\frac{n!+3^n-(n!+2^n)}{(n!+3^n)^{\frac{2}{3}}+\sqrt[3]{(n!+3^n)(n!+2^n)}+(n!+2^n)^{\frac{2}{3}}}=$$

$$=\lim_{n\to\infty}\frac{n^3\,(3^n-2^n)}{(n!+3^n)^{\frac{2}{3}}+\sqrt[3]{(n!+3^n)(n!+2^n)}+(n!+2^n)^{\frac{2}{3}}}$$

Zeshora i zdola vytkneme nejsilnější člen dle růstové škály - tedy nahoře 3^n a dole zvnitřku $n!$.

$$\lim_{n\to\infty}\frac{n^3\,(3^n-2^n)}{(n!+3^n)^{\frac{2}{3}}+\sqrt[3]{(n!+3^n)(n!+2^n)}+(n!+2^n)^{\frac{2}{3}}}=$$

$$=\lim_{n\to\infty}\frac{3^n\cdot n^3}{(n!)^{\frac{2}{3}}}\cdot\frac{1-\frac{2^n}{3^n}}{\left(1+\frac{3^n}{n!}\right)^{\frac{2}{3}}+\sqrt[3]{\left(1+\frac{3^n}{n!}\right)\left(1+\frac{2^n}{n!}\right)}+\left(1+\frac{2^n}{n!}\right)^{\frac{2}{3}}}$$

Přičemž platí od určitého $n_0\in\mathbb{N}$ (neb $n^3\ll 3^n$):

$$0\le\frac{3^n\cdot n^3}{(n!)^{\frac{2}{3}}}\le\frac{3^n\cdot 3^n}{(n!)^{\frac{2}{3}}}=\frac{9^n}{(n!)^{\frac{2}{3}}}=\frac{\left((9^n)^{\frac{3}{2}}\right)^{\frac{2}{3}}}{(n!)^{\frac{2}{3}}}=\frac{\left(\left(9^{\frac{3}{2}}\right)^n\right)^{\frac{2}{3}}}{(n!)^{\frac{2}{3}}}=\left(\frac{27^n}{n!}\right)^{\frac{2}{3}}\overset{\text{r.š. + o posl. s kladnými členy}}{\to}0$$

Ze dvou strážníků tedy plyne, že $\lim_{n\to\infty} \frac{3^n \cdot n^3}{(n!)^{\frac{2}{3}}} = 0$. Pak platí:

$$\lim_{n\to\infty} \frac{3^n \cdot n^3}{(n!)^{\frac{2}{3}}} \cdot \frac{1-\frac{2^n}{3^n}}{\left(1+\frac{3^n}{n!}\right)^{\frac{2}{3}} + \sqrt[3]{\left(1+\frac{3^n}{n!}\right)\left(1+\frac{2^n}{n!}\right)} + \left(1+\frac{2^n}{n!}\right)^{\frac{2}{3}}} =$$
$$\overset{\text{VoAL}}{=} 0 \cdot \frac{1-0}{(1+0)^{\frac{2}{3}} + \sqrt[3]{(1+0)(1+0)} + (1+0)^{\frac{2}{3}}} = 0$$

Příklad 1 (f) $\lim_{n\to\infty} \frac{(n+\sin n)^7 - \left(n+\sqrt{n}\right)^7}{n^2\sqrt{n^7+7}} \cdot \arctan\frac{1}{n}$

Poznámka: $\lim_{n\to\infty} n \cdot \arctan\frac{1}{n} = 1$ plyne z Heineho věty, známé limity a věty o limitě složené funkce (vše bude probíráno v rámci limit funkcí).

$$\lim_{n\to\infty} \frac{(n+\sin n)^7 - (n+\sqrt{n})^7}{n^2\sqrt{n^7+7}} \cdot \arctan\frac{1}{n} =$$
$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{6}\binom{7}{k} n^k \left((\sin n)^{7-k} - (\sqrt{n})^{7-k}\right)}{n^{2+\frac{7}{2}}\sqrt{1+\frac{7}{n^7}}} \cdot \arctan\frac{1}{n} =$$
$$= \lim_{n\to\infty} \frac{n^6 \sum_{k=0}^{6}\binom{7}{k} n^{k-6} \left((\sin n)^{7-k} - (\sqrt{n})^{7-k}\right)}{n^{2+\frac{7}{2}}\sqrt{1+\frac{7}{n^7}}} \cdot \arctan\frac{1}{n} =$$
$$= \lim_{n\to\infty} \frac{n^{\frac{1}{2}} \sum_{k=0}^{6}\binom{7}{k} n^{k-6} \left((\sin n)^{7-k} - (\sqrt{n})^{7-k}\right)}{\sqrt{1+\frac{7}{n^7}}} \cdot \arctan\frac{1}{n} =$$
$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{6}\binom{7}{k} n^{k-5-\frac{1}{2}} \left((\sin n)^{7-k} - (\sqrt{n})^{7-k}\right)}{\sqrt{1+\frac{7}{n^7}}} \cdot \arctan\frac{1}{n} =$$
$$\overset{\text{VoAL}}{=} \frac{1}{1} \cdot \lim_{n\to\infty} \sum_{k=0}^{6} \binom{7}{k} n^{k-5-\frac{1}{2}} \left((\sin n)^{7-k} - \left(\sqrt{n}\right)^{7-k}\right) \cdot \arctan\frac{1}{n}$$

Přičemž platí, že $\lim_{n\to\infty} n^{k-5-\frac{1}{2}} (\sin n)^{7-k} \cdot \arctan\frac{1}{n} = 0$ pro $k-5-\frac{1}{2} < 0$ (limita typu 0·om.). Tedy pro $k=6$ musíme příslušný sčítanec v počítané limitě nechat. Podobně $\lim_{n\to\infty} n^{k-5-\frac{1}{2}} \cdot (\sqrt{n})^{7-k} \cdot \arctan\frac{1}{n} = \lim_{n\to\infty} n^{k-5-\frac{1}{2}+\frac{7}{2}-\frac{k}{2}} \cdot \arctan\frac{1}{n} = \lim_{n\to\infty} n^{\frac{k}{2}-2} \cdot \arctan\frac{1}{n} = 0$ pro $\frac{k}{2} - 2 < 0$. Pro zbylá k - tj. pro $k \in \{5,6\}$ musíme vyřešit zvlášť.

Dostáváme tedy, že:

$$\lim_{n\to\infty} \sum_{k=0}^{6} \binom{7}{k} n^{k-5-\frac{1}{2}} \left((\sin n)^{7-k} - \left(\sqrt{n}\right)^{7-k}\right) \cdot \arctan\frac{1}{n} =$$
$$\overset{\text{VoAL}}{=} \lim_{n\to\infty} \left(\binom{7}{6}\sqrt{n}\sin n - \binom{7}{5}\sqrt{n} - \binom{7}{6} n\right) \cdot \arctan\frac{1}{n} =$$
$$\overset{\text{VoAL}}{=} \lim_{n\to\infty} \left(7\frac{\sin n}{\sqrt{n}} - 21\frac{1}{\sqrt{n}} - 7\right) \cdot n \cdot \arctan\frac{1}{n} \overset{\text{VoAL}}{=} (7\cdot 0 - 21 \cdot 0 - 7)\cdot 1 = -7$$

Poslední rovnost plyne z věty o limitě typu nula krát omezená a faktu, že $\lim_{n\to\infty} n \cdot \arctan \frac{1}{n} = 1$.

Příklad 1 (g) $\lim_{n\to\infty} \left(\sqrt[3]{3^n+2^n} - \sqrt[3]{3^n+n}\right) \cdot \left(\frac{3^n+2^n}{3^n+n} + (-1)^n n^2\right)$

$$\lim_{n\to\infty} \left(\sqrt[3]{3^n+2^n} - \sqrt[3]{3^n+n}\right) \cdot \left(\frac{3^n+2^n}{3^n+n} + (-1)^n n^2\right) =$$

$$= \lim_{n\to\infty} \frac{3^n+2^n-3^n-n}{\left(\sqrt[3]{3^n+2^n}\right)^2 + \sqrt[3]{(3^n+2^n)(3^n+n)} + \left(\sqrt[3]{3^n+n}\right)^2} \cdot \left(\frac{3^n\left(1+\left(\frac{2}{3}\right)^n\right)}{3^n\left(1+\frac{n}{3^n}\right)} + (-1)^n n^2\right) =$$

$$= \lim_{n\to\infty} \frac{2^n\left(1-\frac{n}{2^n}\right)}{\left(3^{\frac{2}{3}}\right)^n\left(\left(\sqrt[3]{1+\left(\frac{2}{3}\right)^n}\right)^2 + \sqrt[3]{\left(1+\left(\frac{2}{3}\right)^n\right)\left(1+\frac{n}{3^n}\right)} + \left(\sqrt[3]{1+\frac{n}{3^n}}\right)^2\right)} \cdot \left(\frac{1+\left(\frac{2}{3}\right)^n}{1+\frac{n}{3^n}} + (-1)^n n^2\right) =$$

$$= \lim_{n\to\infty} \left(\frac{2}{\sqrt[3]{9}}\right)^n \frac{1-\frac{n}{2^n}}{\left(\sqrt[3]{1+\left(\frac{2}{3}\right)^n}\right)^2 + \sqrt[3]{\left(1+\left(\frac{2}{3}\right)^n\right)\left(1+\frac{n}{3^n}\right)} + \left(\sqrt[3]{1+\frac{n}{3^n}}\right)^2} \cdot \left(\frac{1+\left(\frac{2}{3}\right)^n}{1+\frac{n}{3^n}} + (-1)^n n^2\right) =$$

$$\overset{\text{VoAL + r.š.}}{=} \frac{1-0}{1+1+1} \lim_{n\to\infty} \left(\frac{2}{\sqrt[3]{9}}\right)^n \cdot \left(\frac{1+\left(\frac{2}{3}\right)^n}{1+\frac{n}{3^n}} + (-1)^n n^2\right) =$$

$$\overset{\text{VoAL + r.š.}}{=} \frac{1}{3} \cdot \left(0 \cdot \frac{1+0}{1+0} + \lim_{n\to\infty} (-1)^n \frac{n^2}{\left(\frac{\sqrt[3]{9}}{2}\right)^n}\right) \overset{\text{VoAL + r.š}}{=} 0$$

Neb: $\sqrt[3]{9} > 2$, tedy $\frac{\sqrt[3]{9}}{2} > 1$.

Příklad 2

Vzorové řešení doc. Johanise. (odkaz)